Mathematics Olympiad - Final Stage

- Section 1 consists of 8 problems. No solutions are required, only the answers in allocated spaces are to be filled. Every correct answer is worth of 2 point.
- Section 2 consists of 2 work-out problems of 4 and 5 points. The solutions must be written clearly in this booklet.
- White sheets provided are for draft, which is not to be graded.
- Duration is 1 hour. Pick your own strategy to win.
- Write your full name below and do not open the booklet until instructed to do so. Good luck!

FULL NAME:

Do NOT write in this table

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	P9	P10	$\sum$
2	2	2	2	2	2	2	2	4	5	25

February 26th, 2021

1 Short-Answer Questions

Q1. Solve the congruence $4x \equiv 5 \pmod 9$

(Note: $a \equiv b \pmod n$ means $b - a$ is divisible by n.)

Answer:

Q2. Find $\gcd(2n + 1, 3n + 2)$, where n is any positive integer.

(Note: gcd=greatest common divisor.)

Answer:

Q3. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

Answer:

Q4. Simplify $\sum_{k=1}^{n} \binom{n}{k}^2$.

(Note: here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.)

Answer:

Q5. A young man is climbing the stairs and it is known that he can take one stair or two stairs at a time. In how many ways can this person climb a flight of eight stairs?

Answer:

Q6. Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

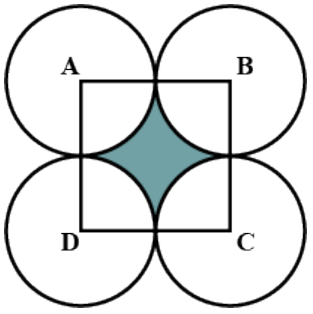
for $n \geq 2$ together with initial values $a_0 = 1, a_1 = 0$.

Answer:

Q7. Quadrilateral $ABCD$ has right angles at A and D. A circle of radius 10 fits neatly inside the quadrilateral and touches all four sides. The length of edge BC is 24. The midpoint of edge AD is called E and the midpoint of edge BC is called F. What is the length of EF?

Answer:

Q8. In figure, $ABCD$ is a square of side 14 cm. With centres A, B, C and D, four circles are drawn such that each circle touch externally two of the remaining three circles. Find the area of the shaded region.



Answer:

2 Work-Out Problems

P9. (4 points) Points A, B, and C lie on a circle with center M. The reflection of point M in the line AB lies inside triangle ABC and is the intersection of the angular bisectors of angles A and B. (The angular bisector of an angle is the line that divides the angle into two equal angles.) Line AM intersects the circle again in point D. Show that $|CA| \cdot |CD| = |AB| \cdot |AM|$.

P10. (5 points) Show that every integer greater than 11 is the sum of two composite integers.

Solutions of Work-Out Problems